The Variational Iteration Method for Finding Exact Solution of Nonlinear Gas Dynamics Equations

Hossein Jafari[a], Ch. Chun[b], and C. M. Khalique[c]

[a] Department of Mathematics, University of Mazandaran, Babolsar, Iran
[b] Department of Mathematics, Sungkyunkwan University Suwon, Republic of Korea
[c] International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, Republic of South Africa

Reprint requests to H. J.; E-mail: jafari_h@math.com

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The variational iteration method (VIM) proposed by Ji-Huan He is a new analytical method to solve nonlinear equations. In this paper, a modified VIM is introduced to accelerate the convergence of VIM and it is applied for finding exact analytical solutions of nonlinear gas dynamics equation.

Key words: Variational Iteration Method; Gas Dynamics Equation; Analytical Solution; Nonlinear Equations.
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1. Introduction

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand give rise to rounding off errors. Recently, the variational iteration method (VIM), introduced by He (see [1, 2] and references therein), which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proved successful in deriving analytical solutions of linear and nonlinear differential equations. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. He has applied this method for obtaining analytical solutions of autonomous ordinary differential equation, nonlinear partial differential equations with variable coefficients, and integro-differential equations. The variational iteration method was successfully employed by various authors. For example, the VIM was applied to the nonlinear Boltzmann equation [3], to Burger's and coupled Burger's equations [4], to the eikonal partial differential equation [5], to parabolic integro-differential equations arising in heat conduction in materials with memory [6], to coupled Korteweg-de Vries (KdV) and Boussinesq-like B(m,n) equations [7], to Sawada-Kotera equations [8], to modified Camassa-Holm and Degasperis-Procesi equations [9], to KdV, K(2,2), Burgers, and cubic Boussinesq equations [10] and to KdV-Burgers and Sharma-Tasso-Olver equations [11].

In the present paper, the VIM is employed to solve the following type of partial differential equation:

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial(u^2)}{\partial x}=u(1-u)+f(x,t), \quad 0\le x\le 1, \quad t>0, \tag{1}$$

and we consider a modified VIM to accelerate the convergence of the VIM. Equation (1) is known as the nonhomogeneous gas dynamics equation [12].

2. He's Variational Iteration Method

For the purpose of illustration of the methodology to the proposed method, using variational iteration method, we begin by considering a differential equation in the formal form,

$$Lu+Nu=f(x,t), \tag{2}$$

where L is a linear operator, N a nonlinear operator, and $f(x,t)$ is the source inhomogeneous term. According to the variational iteration method, we can construct a correction functional for (2) as follows:

$$u_{n+1}(x,t)= u_n(x,t)+\int_0^t \lambda\,\{Lu_n(\tau)+N\tilde{u}_n(\tau)-f(\tau)\}\,\mathrm{d}\tau, \quad n\ge 0, \tag{3}$$

where λ is a general Lagrangian multiplier [13], which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, and $\tilde{u}_n$ is considered as a restricted variation [13], i. e., $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts [14]. The successive approximations $u_n(x,t), n \geq 0$, of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{4}$$

For the convergence of the above method we refer the reader to [15].

3. Applications with Modified VIM

We often run into the problems, such as the ones considered in this contribution, whose successive approximate solutions of VIM converge to its exact solution relatively slowly. In the following sections, a modified VIM is considered to deal with these situations.

3.1. Homogeneous Gas Dynamics Equation

To apply the VIM, first we rewrite (1) with $f(x,t) = 0$ in the form

$$Lu + Nu = 0, \tag{5}$$

where the notations $Lu = \frac{\partial u}{\partial t}$, $Nu = 1/2\frac{\partial(u^2)}{\partial x} - u + u^2$, symbolize the linear and nonlinear terms, respectively. The correction functional for (5) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ \frac{\partial}{\partial\tau}(u_n) + N(\tilde{u}_n) \right\} \mathrm{d}\tau, \quad n \geq 0. \tag{6}$$

By taking the variation with respect to the independent variable u_n, noticing that $\delta N(\tilde{u}_n(0)) = 0$, we get

$$\begin{aligned} &\delta u_{n+1}(x,t) \\ &\quad = \delta u_n(x,t) + \delta \int_0^t \lambda \left\{ \frac{\partial}{\partial\tau}(u_n) + N(\tilde{u}_n) \right\} \mathrm{d}\tau \\ &\quad = \delta u_n(x,t) + \lambda\delta u_n \,|_{\tau=t} - \int_0^t \lambda' \delta u_n \mathrm{d}\tau \\ &\quad = 0. \end{aligned} \tag{7}$$

This yields the stationary conditions

$$1 + \lambda(\tau) = 0, \tag{8}$$

$$\lambda'(\tau) \,|_{\tau=t} = 0. \tag{9}$$

Equation (8) is called Lagrange-Euler equation, and (9) natural boundary condition. The Lagrange multiplier can be identified as $\lambda = -1$, and the following variational iteration formula can be obtained:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial}{\partial\tau}(u_n) + u_n \frac{\partial}{\partial x}(u_n) - u_n + u_n^2 \right\} \mathrm{d}\tau, \quad n \geq 0. \tag{10}$$

We start with an initial approximation $u_0(x,t) = \mathrm{e}^{-x}$, and by means of the iteration formula (10), we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) &= \mathrm{e}^{-x}(1+t), \\ u_2(x,t) &= \mathrm{e}^{-x}(1+t+\frac{t^2}{2}), \\ u_3(x,t) &= \mathrm{e}^{-x}(1+t+\frac{t^2}{2}+\frac{t^3}{3!}), \\ u_4(x,t) &= \mathrm{e}^{-x}(1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\frac{t^4}{4!}), \\ &\vdots \end{aligned} \tag{11}$$

This gives the exact solution of (5) by

$$u(x,t) = \mathrm{e}^{t-x}, \tag{12}$$

obtained upon using the Taylor expansion for e^t.

From the above solution procedure, we can see clearly that the approximate solutions converge to its exact solution e^{t-x} relatively slowly due to the approximate identification of the multiplier.

To accelerate the convergence, we apply restricted variations to a few nonlinear terms, therefore, we rewrite (1) in the form

$$Lu + Nu = 0, \tag{13}$$

where the notations $Lu = \frac{\partial u}{\partial t} - u$, $Nu = u^2$, symbolize the linear and nonlinear terms, respectively. The correction functional for (13) reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ \frac{\partial}{\partial\tau}(u_n) - u_n + N(\tilde{u}_n) \right\} \mathrm{d}\tau, \quad n \geq 0. \tag{14}$$

Making the above correction functional stationary with respect to u_n, noticing that $\delta N(\tilde{u}_n)(0) = 0$, we get

$$\begin{aligned}&\delta u_{n+1}(x,t)\\&= \delta u_n(x,t) + \delta \int_0^t \lambda \left\{ \frac{\partial}{\partial \tau}(u_n) - u_n + N(\tilde{u}_n) \right\} \mathrm{d}\tau\\&= \delta u_n(x,t) + \lambda \delta u_n \,|_{\tau=t}\\&\quad + \int_0^t \left\{ -\lambda' \delta u_n - \lambda \delta u_n \right\} \mathrm{d}\tau = 0.\end{aligned} \tag{15}$$

Hence, we obtain the Euler-Lagrange equation

$$1 + \lambda \,|_{\tau=t} = 0 \tag{16}$$

and the natural boundary condition

$$\lambda' + \lambda \,|_{\tau=t} = 0. \tag{17}$$

We, therefore, identify the Lagrange multiplier in the form

$$\lambda = -\mathrm{e}^{t-\tau}. \tag{18}$$

Substituting this Lagrange multiplier into (14) results in the following iteration formulation:

$$\begin{aligned}&u_{n+1}(x,t) =\\&u_n(x,t) - \int_0^t \mathrm{e}^{t-\tau} \left\{ \frac{\partial}{\partial \tau}(u_n) + u_n \frac{\partial}{\partial x}(u_n) - u_n + u_n^2 \right\} \mathrm{d}\tau,\\&n \geq 0.\end{aligned} \tag{19}$$

If we begin with $u_0(x,t) = \mathrm{e}^{-x}$, by means of the above iteration formula (19), we can obtain directly the other components as

$$\begin{aligned}u_1(x,t) &= \mathrm{e}^{t-x},\\u_2(x,t) &= \mathrm{e}^{t-x},\\u_3(x,t) &= \mathrm{e}^{t-x},\\&\vdots\end{aligned} \tag{20}$$

So, it can be seen clearly that the approximations obtained from (19) converge to its exact solution more fast than those obtained from the iteration formula (10).

3.2. Non-Homogeneous Gas Dynamics Equation

We now rewrite (1) with $f(x,t) = -\mathrm{e}^{t-x}$ in the form

$$Lu + Nu = 0, \tag{21}$$

where the notations $Lu = \frac{\partial u}{\partial t} - u$, $Nu = u^2 + \mathrm{e}^{t-x}$, symbolize the linear and nonlinear terms, respectively. The correction functional for (21) reads

$$\begin{aligned}&u_{n+1}(x,t) =\\&u_n(x,t) + \int_0^t \lambda \left\{ \frac{\partial}{\partial \tau}(u_n) - u_n + N(\tilde{u}_n) \right\} \mathrm{d}\tau,\\&n \geq 0.\end{aligned} \tag{22}$$

By taking variation with respect to the independent variable u_n and noticing that $\delta N(\tilde{u}_n)(0) = 0$, we get

$$\begin{aligned}&\delta u_{n+1}(x,t) =\\&\delta u_n(x,t) + \delta \int_0^t \lambda \left\{ \frac{\partial}{\partial \tau}(u_n) - u_n + N(\tilde{u}_n) \right\} \mathrm{d}\tau\\&= \delta u_n(x,t) + \lambda \delta u_n \,|_{\tau=t}\\&\quad + \int_0^t \left\{ -\lambda' \delta u_n - \lambda' \delta u_n \right\} \mathrm{d}\tau = 0.\end{aligned} \tag{23}$$

This yields the stationary conditions

$$1 + \lambda = 0, \tag{24}$$

$$\lambda' + \lambda \,|_{\tau=t} = 0. \tag{25}$$

The Lagrange multiplier can be easily identified as $\lambda = -\mathrm{e}^{t-\tau}$, and the following variational iteration formula can be obtained:

$$\begin{aligned}&u_{n+1}(x,t) = u_n(x,t)\\&- \int_0^t \mathrm{e}^{t-\tau} \left\{ \frac{\partial}{\partial \tau}(u_n) + u_n \frac{\partial}{\partial x}(u_n) - u_n + u_n^2 + \mathrm{e}^{t-x} \right\} \mathrm{d}\tau,\\&n \geq 0.\end{aligned} \tag{26}$$

We start with an initial approximation $u_0(x,t) = 1 - \mathrm{e}^{-x}$, by means of the above iteration formula (26), we can obtain directly the other components as

$$\begin{aligned}u_1(x,t) &= 1 - \mathrm{e}^{-x} + \mathrm{e}^{t-x}(-t),\\u_2(x,t) &= 1 - \mathrm{e}^{-x} + \mathrm{e}^{t-x}\left(-t + \frac{t^2}{2}\right),\\u_3(x,t) &= 1 - \mathrm{e}^{-x} + \mathrm{e}^{t-x}\left(-t + \frac{t^2}{2} - \frac{t^3}{3!}\right),\\u_4(x,t) &= 1 - \mathrm{e}^{-x} + \mathrm{e}^{t-x}\left(-t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!}\right),\\&\vdots\end{aligned} \tag{27}$$

This gives the exact solution of (21) by

$$u(x,t) = 1 - \mathrm{e}^{t-x}, \tag{28}$$

obtained upon using the Taylor expansion for $\mathrm{e}^{-t} - 1$.

4. Conclusion

In this paper, the variational iteration method has been successfully applied to find solutions of some gas dynamics equations. The solution obtained by the variational iteration method is an infinite power series for which, with appropriate initial condition, can be expressed the exact solution in a closed form. The results presented in this contribution show that the variational iteration method is a powerful mathematical tool for solving gas dynamics equations, it is also a promising method to solve some other nonlinear equations.

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